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Quantization of the open string on plane-wave limits of $dS_n \times S^n$ and non-commutativity outside branes

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Abstract

The open string on the plane-wave limit of $dS_n \times S^n$ with constant B_2 and dilaton background fields is canonically quantized. This entails solving the classical equations of motion for the string, computing the symplectic form, and defining from its inverse the canonical commutation relations. Canonical quantization is proved to be perfectly suited for this task, since the symplectic form is unambiguously defined and non-singular. The string position and the string momentum operators are shown to satisfy equal-time canonical commutation relations. Noticeably the string position operators define non-commutative spaces for all values of the string world-sheet parameter σ, thus extending non-commutativity outside the branes on which the string endpoints may be assumed to move. The Minkowski space–time limit is smooth and reproduces the results in the literature, in particular non-commutativity gets confined to the endpoints.

1. Introduction

Solutions to the Einstein equations in general relativity have been known for a long time to have plane waves as limits [1]. These limits, known as Penrose limits, give a plane wave space–time approximation for the full space–time along a null geodesic. This observation led in the sixties and seventies to a detailed study of the geometric properties of plane-wave metrics and of matter fields defined on them [2]. Already within string theory, it soon became clear that higher-dimensional plane waves give exact solutions to string theory, provided the Kalb–Ramond and dilaton fields satisfy certain conditions [3,4]. The generalization of the Penrose

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limiting procedure relating higher-dimensional plane waves with more complicated solutions to string theory [5] further triggered the interest in such space–times.

By now, there is a very extensive literature on plane waves in string theory. Motivated by the fact that $AdS_4 \times S^7$ and $AdS_7 \times S^4$ are solutions to M-theory and $AdS_5 \times S^5$ is a solution of IIB supergravity, and by the AdS/CFT correspondence, special attention has been given the Penrose limit [6–8]

$$AdS_k \times S^n \quad pp\text{-limit:} \quad ds^2 = -dx^+ dx^- - m^2 \mathbf{x}^2_{k+n-2} \big(dx^+\big)^2 + d\mathbf{x}^2_{k+n-2} \tag{1.1}$$

of $AdS_k \times S^n$ spaces. Two milestones in this regard are (i) the quantization [9] of the R–R sector of the closed superstring on this background for $k = n = 5$, and (ii) the derivation of its spectrum from that of $U(N)$ $\mathcal{N} = 4$ super-Yang–Mills theory [10]. The interest has extended also to type IIB superstring models in 6 dimensions [11] describing generalizations of the Nappi–Witten model. As a matter of fact, the Nappi–Witten model [12] is itself the Penrose limit of $AdS_2 \times S^2$. There has been as well interest on strings on 4-dimensional homogenous plane-wave backgrounds [13]. These have the form (1.1) with m^2 replaced by a function $C|x^+|^{-2}$ and, for different values of the constant C, occur as the Penrose limit of FRW metrics, near horizon regions of Dp-brane backgrounds and fundamental strings backgrounds [8,14].

In this paper we consider quantization of the open string on the Penrose limit of $dS_n \times S^n$ with non-zero constant 2-form B_2. To date, no background p-forms have been found that support $dS_n \times S^n$ as a solution to IIB supergravity. Yet there are indications that de Sitter space may occur in type IIA theories [15]. In any case, the Penrose limit of $dS_n \times S^n$ is an exact solution of string theory in the critical dimension [4]. There are other motivations for taking de Sitter space–time: its "apparent" simplicity when it comes to quantum gravity [16], the dS/CFT correspondence [17] and the fact that the non-existence of a positive conserved energy indicates that there cannot be unbroken supersymmetry, so it seems a good starting point to go down in the number of supersymmetries. The motivation for taking $B_2 \neq 0$ comes from an interest in understanding non-commutativity in relation with gravity. As is well known, string theory gives explicit realizations of non-commutative spaces. The simplest example is provided by an open string in Minkowski space–time with endpoints moving on a D-brane on which a magnetic field is defined: upon quantization, the string position operators generate a non-commutative space along the brane [18–20]. Since non-commutativity is postulated as a candidate to reconcile quantum mechanics with general relativity [21], and the low energy limit of string theory includes general relativity, it seems natural to explore the non-commutativity/gravity connection within string theory. One way to push forward this approach is to examine non-commutativity for plane wave backgrounds. As a matter of fact, this program has already started for the open string on plane-wave limits of $AdS_n \times S^n$. In 10 dimensions with a constant non-zero B_2 in Ref. [22], and in 4 dimensions with a Nappi–Witten 2-form in [23]. In both instances, the string endpoints define non-commutative spaces. Here we investigate non-commutativity for the Penrose limit of $dS_n \times S^n$.

More precisely, we will quantize the open string interacting through a plane-wave metric

$$ds^2 = -dx^+ dx^- + m^2\big[\big(x^1\big)^2 - \big(x^2\big)^2\big]\big(dx^+\big)^2 + \sum_{i=1}^{2}\big(dx^i\big)^2 + \sum_{a=3}^{D-2}\big(dx^a\big)^2 \tag{1.2}$$

and constant antisymmetric and dilaton fields

$$B_{ij} = \epsilon_{ij} B, \qquad B_{ab} = 0, \qquad \Phi = \Phi_0. \tag{1.3}$$

It will come out that the string position operators $X^1(\tau,\sigma)$ and $X^2(\tau,\sigma')$ do not commute for arbitrary σ and σ'. This is in contrast with the results available so far for open strings on $AdS_n \times S^n$ plane-wave limits supported by a non-zero B_2 [19,22,23], for which non-commutativity is restricted to the brane manifold on which the string endpoints move. Our results are consistent with those in the literature for Minkowski space–time [19], since the latter are recovered in the limit $m \to 0$; in particular non-commutativity gets confined to the string endpoints.

We will work in light-cone and conformal gauges. The paper is organized as follows. In Section 2 we derive the equations of motion for the classical string and solve them. The solution turns out to be an infinite sum over modes, with a highly non-trivial dependence on the parameter m. As compared to the open string in Minkowski space–time, two important differences are encountered. The first one is that the string has a finite number of non-oscillating degrees of freedom associated to modes exponentially growing and decaying in τ. The second one is that the string total momentum is not an independent degree of freedom but receives contributions from all the modes. In Section 3 the string is canonically quantized. This is done by calculating the symplectic form and then using it to find the commutation relations for the operators associated to all the string modes. The symplectic form is unambiguous and non-singular, not being necessary to provide additional constraints or to modify its definition so as to fix the commutators. As a check it is shown that the string momentum and the string position operators satisfy equal-time canonical commutation relations. Section 4 shows that the string position operators X^1 and X^2 do not commute for arbitrary values of σ and σ', thus defining non-commutative waves fronts. In Section 5 we find the eigenstates and the spectrum of the Hamiltonian. Section 6 contains our conclusions. We have included Appendices A and B with some of the details of the calculations of Sections 2 and 4.

2. The classical string

Due to its length, this section is divided into five parts. In the first one, we study the background metric (1.2). Section 2.2 contains the derivation of the equations of motion and of the boundary conditions for the classical open string in the background (1.2)–(1.3). The equations of motion are solved in Section 2.3, where expressions for the string coordinates as sums over modes ready to be quantized are found. Section 2.4 presents a brief discussion of the string center of mass coordinates and the string total momentum. Finally, in Section 2.5 we discuss the case $m^2\kappa^2 \ll 1$, with $X^+ = \kappa\tau$ the light-cone gauge condition.

2.1. The background as the Penrose limit of $dS_n \times S^n$

The metric (1.2) is the Penrose limit of $dS_2 \times S^2 \times \mathrm{E}^{D-4}$, with E^{D-4} Euclidean space in $D-4$ dimensions. Although well known, let us very briefly check this point. Consider k-dimensional de Sitter space–time times an n-sphere, $dS_k \times S^n$, both of radius ℓ. Its metric can be written as

$$ds^2 = \ell^2\left[-\left(1-\rho^2\right)dt^2 + \frac{d\rho^2}{1-\rho^2} + \rho^2\, d\Omega^2_{k-2} + \left(1-r^2\right)d\chi^2 + \frac{dr^2}{1-r^2} + r^2\, d\Omega'^2_{n-2}\right], \tag{2.1}$$

where $d\Omega^2_{k-2}$ and $d\Omega'^2_{n-2}$ are the round metrics on the unit $(k-2)$ and $(n-2)$-spheres. Consider now, as in the anti-de Sitter case [10], the trajectory along χ in the vicinity of $\rho = r = 0$. Making the changes $u^{\pm} = t \pm \chi$, rescaling

$$u^+ = x^+, \quad u^- = \frac{x^-}{\ell^2}, \quad \rho = \frac{\bar{\rho}}{\ell}, \quad r = \frac{\bar{r}}{\ell} \quad \text{with } \ell \to \infty, \tag{2.2}$$

and introducing a mass scale $x^+ \to 2mx^+$, $x^- \to x^-/2m$, one arrives at

$$dS_k \times S^n \quad pp\text{-limit:} \quad ds^2_{pp} = -dx^+ dx^- + m^2\big(\mathbf{x}^2_{k-1} - \mathbf{y}^2_{n-1}\big)\big(dx^+\big)^2 + d\mathbf{x}^2_{k-1} + d\mathbf{y}^2_{n-1}. \tag{2.3}$$

Here Cartesian coordinates $\mathbf{x}_{k-1} = (\bar{\rho}, \Omega_{k-2})$ and $\mathbf{y}_{n-1} = (\bar{r}, \Omega_{n-2})$ have been introduced. Backgrounds

$$ds^2 = ds^2_{pp} + ds^2\left(\mathrm{E}^{D-n-k}\right), \qquad H_3 = dB_2 = A_{ij}\big(x^+\big)\, dx^+ \wedge dx^i \wedge dy^j$$

are solutions to all orders in α' for the bosonic/fermionic string in $D = 26/10$ provided A_{ij} satisfies the condition [4]

$$4m^2(n-k) = A_{ij}A^{ij}.$$

H_3 vanishes for $k = n$, in which case one may take $B_2 = B_{ij}\, dx^i \wedge dy^j$, with B_{ij} constant. The metric (1.2) is recovered for $k = n = 2$ and is non-singular, meaning it is geodesically complete. The results in this paper are trivially extended to the case $k = n = 5$.

It is important to note the positive sign in front of $\mathbf{x}^2_{k-1}$ in the metric coefficient g_{++} in Eq. (2.3). This has its origin in the fact that we have started with de Sitter space–time, rather than anti-de Sitter, and implies that the metric (2.3) does not admit a conserved positive energy. To understand this we recall that in de Sitter space there is no positive conserved energy since there is no generator of its isometry group, $SO(1, d)$, which is timelike everywhere. In the coordinates (2.1), the generator $\partial/\partial t$ is timelike for $\rho < 1$, but vanishes at the event horizon $\rho = 1$. Hence, $\partial/\partial t$ and its associated Hamiltonian can only be used to define time evolution in the region $0 \leqslant \rho \leqslant 1$ within the event horizon. Upon forming $dS_n \times S^n$ and taking the Penrose limit, this implies that for the metric (2.3) the sign of the energy depends on the sign of $\mathbf{x}^2_{k-1} - \mathbf{y}^2_{n-1}$. This is a property of the background considered.

2.2. *Classical action, field equations and momenta*

Our starting point is the bosonic part of the classical action

$$S = \frac{1}{4\pi\alpha'} \int d\tau\, d\sigma\, \left(\sqrt{-\gamma}\gamma^{rs} G_{\mu\nu}\partial_r X^{\mu}\partial_s X^{\nu} + \epsilon^{rs} B_{\mu\nu}\partial_r X^{\mu}\partial_s X^{\nu} + \alpha'\sqrt{-\gamma}\, R\Phi\right)$$

for the open string on the D-dimensional background $G_{\mu\nu}(X)$, $B_{\mu\nu}(X)$, $\Phi(X)$ in Eqs. (1.2)–(1.3). Greek letters $\mu, \nu, \ldots$ denote space–time indices, while lower case letters $r, s, \ldots$ from the end of the Roman alphabet denote world-sheet indices. Here γ_{rs} is the metric on the string world-sheet, R its scalar curvature and ϵ^{rs} is defined by $\epsilon^{01} = 1$. As usual the world-sheet coordinates τ and σ take values on the intervals $-\infty < \tau < \infty$ and $0 \leqslant \sigma \leqslant \pi$. We are using units in which string coordinates have dimensions of length and τ, σ are dimensionless. From now on we will use capital case letters X's for the string coordinates.

If wished, the string endpoints may be assumed to lie on a Dp-brane on which a magnetic field F_{ij} lives.[1] This amounts to adding to the action a term

$$\delta S = \frac{1}{2\pi\alpha'}\int d\tau\, A_i \partial_\tau X^i\Big|_{\sigma=0}^{\sigma=\pi},$$

with $A_i(X)$ the $U(1)$ gauge field on the brane. If this term is included in the action, the analysis in this paper goes through with the only difference that the field B_{ij} must be replaced by the Born–Infeld field strength $\mathcal{B}_{ij} = B_{ij} - F_{ij}$, where F_{ij} is the $U(1)$ field strength on the brane.

The string action has three world-sheet symmetries. We will fix one of them by working in light-cone gauge [24]

$$X^+ = \kappa\tau,$$

with κ a parameter with dimensions of length. The other two will be fixed by choosing conformal gauge

$$h^{rs} = \sqrt{-\gamma}\,\gamma^{rs} = \mathrm{diag}(-1,+1).$$

In this gauge, the classical action becomes

$$S = \int d\tau\, L,$$

where the Lagrangian L is given by

$$L = p_-\partial_\tau x^- - \frac{1}{4\pi\alpha'}\int_0^\pi d\sigma\,\big\{m^2\kappa^2\big[\big(X^1\big)^2 - \big(X^2\big)^2\big] + \big(\partial_\tau X^i\big)^2 + \big(\partial_\tau X^a\big)^2$$
$$- \big(\partial_\sigma X^i\big)^2 - \big(\partial_\sigma X^a\big)^2 - 2B\big[\partial_\tau X^1\partial_\sigma X^2 - \partial_\sigma X^1\partial_\tau X^2\big]\big\},$$

with

$$p_- = -\frac{\kappa}{4\alpha'}$$

the momentum conjugate to $x^-(\tau)$, defined [25] as the average over σ at a given τ of $X^-(\tau,\sigma)$

$$x^-(\tau) = \frac{1}{\pi}\int_0^\pi d\sigma\, X^-(\tau,\sigma).$$

Here we have reserved the subscript i for the 1 and 2 directions, while a runs from 3 to $D-2$, a convention that we will follow from now on.

The field equations and boundary conditions are obtained by varying the action with respect to X^i and X^a. They take the form

$$\Box X^1 + m^2\kappa^2 X^1 = 0, \tag{2.4}$$

$$\Box X^2 - m^2\kappa^2 X^2 = 0, \tag{2.5}$$

$$\Box X^a = 0, \tag{2.6}$$

[1] p is 1 for $k=n=2$ in (2.3) and 4 for $k=n=5$.

with $\Box = -\partial_\tau^2 + \partial_\sigma^2$ the 2-dimensional d'Alambertian, and

$$\partial_\sigma X^1 - B\partial_\tau X^2\big|_{\sigma=0,\pi} = 0, \tag{2.7}$$

$$\partial_\sigma X^2 + B\partial_\tau X^1\big|_{\sigma=0,\pi} = 0, \tag{2.8}$$

$$\partial_\sigma X^a\big|_{\sigma=0,\pi} = 0. \tag{2.9}$$

To quantize the theory we will need the momenta. In our case, these are given by

$$p_- = -\frac{\kappa}{4\alpha'}, \tag{2.10}$$

$$P_i = \frac{1}{2\pi\alpha'}\big(\partial_\tau X^i - B\epsilon_{ij}\partial_\sigma X^j\big), \tag{2.11}$$

$$P_a = \frac{1}{2\pi\alpha'}\partial_\tau X^a. \tag{2.12}$$

In terms of them, the Lagrangian L can be written as

$$L = -p_-\partial_\tau x^- + \int_0^\pi d\sigma\,\big[-\big(P_i\partial_\tau X^i + P_a\partial_\tau X^a\big) + \mathcal{H}\big],$$

where the Hamiltonian density $\mathcal{H}$ has the form

$$\begin{aligned} 4\pi\alpha'\mathcal{H} = {} & \big(2\pi\alpha' P_i + B\epsilon_{ij}\partial_\sigma X^j\big)^2 + (2\pi\alpha' P_a)^2 + \big(\partial_\sigma X^i\big)^2 + \big(\partial_\sigma X^a\big)^2 \\ & - m^2\kappa^2\big[\big(X^1\big)^2 - \big(X^2\big)^2\big]. \end{aligned} \tag{2.13}$$

We note that $\mathcal{H}$ is not positive definite because of the negative sign in front of $(X^1)^2$. As explained in Section 2.1, this originates in the fact that in de Sitter space–time there is no positive conserved energy and implies that $\mathcal{H}$ can only be used to account for time evolution in the region where it is non-negative.

2.3. Solution to the classical equations of motion

The solution for X^a is the well-known mode sum

$$X^a(\tau,\sigma) = c_0^a + d_0^a\tau + \sum_{n\neq 0}^{\infty} i\,\frac{c_n^a}{n}\cos n\sigma\, e^{-in\tau}, \tag{2.14}$$

where c_n^a are complex constants of integration (mode amplitudes). Reality of X^a implies that c_0^a and d_0^a are real and that $(c_n^a)^\star = c_{-n}^a$.

The solution for X^1 and X^2 is more involved. To find it we use separation of variables $X^i(\tau,\sigma) = T_i(\tau)S_i(\sigma)$. This gives

$$\frac{\ddot{T}_1}{T_1} = \frac{S_1''}{S_1} - m^2\kappa^2 = -\lambda_1^2,$$

$$\frac{\ddot{T}_2}{T_2} = \frac{S_1''}{S_1} + m^2\kappa^2 = -\lambda_2^2,$$

where the dot and prime indicate differentiation with respect to τ and σ respectively. The boundary conditions (2.7) and (2.8) imply that non-trivial solutions are only possible for $\lambda_1 = \lambda_2$. We therefore set $\lambda := \lambda_1 = \lambda_2$, introduce

$$\alpha = \sqrt{\lambda^2 - m^2\kappa^2}, \qquad \beta = \sqrt{\lambda^2 + m^2\kappa^2} \tag{2.15}$$

and distinguish several cases.

Case 1.: $\lambda = 0$. It is straightforward to see that non-trivial solutions only exist if $m\kappa$ is an integer. In particular, for $m\kappa$ an odd integer the solution reads

$$X^1_{\mathrm{o}}(\tau,\sigma) = \left[a_{\mathrm{o}} + b_{\mathrm{o}}\tau \sinh\left(\frac{m\kappa\pi}{2}\right)\right]\cos(m\kappa\sigma), \tag{2.16}$$

$$X^2_{\mathrm{o}}(\tau,\sigma) = \frac{B}{m\kappa} b_{\mathrm{o}} \cosh\left[m\kappa\left(\frac{\pi}{2} - \sigma\right)\right], \tag{2.17}$$

whereas for $m\kappa$ an even integer the solution takes the form

$$X^1_{\mathrm{e}}(\tau,\sigma) = \left[a_{\mathrm{e}} + b_{\mathrm{e}}\tau \cosh\left(\frac{m\kappa\pi}{2}\right)\right]\cos(m\kappa\sigma), \tag{2.18}$$

$$X^2_{\mathrm{e}}(\tau,\sigma) = \frac{B}{m\kappa} b_{\mathrm{e}} \sinh\left[m\kappa\left(\frac{\pi}{2} - \sigma\right)\right], \tag{2.19}$$

with a_{o}, b_{o} and a_{e}, b_{e} arbitrary constants of integration in every instance.

Case 2.: $\lambda^2 = \pm m^2\kappa^2$. This corresponds to either α or β zero and it is very easy to show that the only solution for X^1 and X^2 is the trivial one.

Case 3.: $\lambda^2 \neq 0, \pm m^2\kappa^2$. Solving then for T_i and S_i and imposing the boundary conditions, it follows that the eigenvalues λ must satisfy the equation

$$\left(\lambda^4 B^4 + \alpha^2\beta^2\right)\sin\alpha\pi \sin\beta\pi - 2\lambda^2 B^2 \alpha\beta(\cos\alpha\pi \cos\beta\pi - 1) = 0. \tag{2.20}$$

Solutions to this equation may occur either because both its terms vanish or because none of them vanishes but their sum does. We therefore consider two subcases:

Subcase 3.1. Both terms in Eq. (2.20) vanish. Since α and β are non-zero, we must have

$$\sin\alpha\pi \sin\beta\pi = \cos\alpha\pi \cos\beta\pi - 1 = 0. \tag{2.21}$$

It is very easy to see then that the modes for X^1 and X^2 have the form

$$X^1_{(k,l)}(\tau,\sigma) = \frac{i}{\lambda}\left(a_{\lambda(k,l)} \frac{\alpha}{B} \cos\beta\sigma + b_{\lambda(k,l)} \sin\beta\sigma\right) e^{-i\lambda\tau}, \tag{2.22}$$

$$X^2_{(k,l)}(\tau,\sigma) = -\left(b_{\lambda(k,l)} \frac{\beta}{\lambda^2 B} \cos\alpha\sigma + a_{\lambda(k,l)} \sin\alpha\sigma\right) e^{-i\lambda\tau}, \tag{2.23}$$

where $a_{\lambda(k,l)}$ and $b_{\lambda(k,l)}$ are arbitrary constants of integration. It follows from Eqs. (2.21) that α and β must be integers and that their difference must be an even integer. Hence we write

$$\alpha = k, \qquad \beta = k + 2l, \tag{2.24}$$

with k and l arbitrary positive integers since $\beta \geqslant \alpha$ and α and β are defined as positive. With this, Eqs. (2.15) imply

$$m^2\kappa^2 = 2l(k+l) > 0, \qquad \lambda = \pm\sqrt{l^2 + (l+k)^2}. \tag{2.25}$$

The first one of these equations states that $m^2\kappa^2$ is an even integer. We thus conclude that for $m^2\kappa^2$ an even integer, there are as many modes of type (2.22)–(2.23) as pairs (k,l) of positive integers solving the equation $m^2\kappa^2 = 2l(k+l)$, which is clearly a finite number.

Subcase 3.2. We now look at solutions λ to Eq. (2.20) such that

$$\sin\alpha\pi\,\sin\beta\pi \neq 0. \tag{2.26}$$

In this case the modes for X^1 and X^2 read

$$X^1_\lambda(\tau,\sigma) = i\,\frac{c_\lambda}{\lambda B}\left(\alpha\cos\beta\sigma + \frac{K_\lambda}{\beta}\sin\beta\sigma\right)e^{-i\lambda\tau}, \tag{2.27}$$

$$X^2_\lambda(\tau,\sigma) = -\left(\frac{K_\lambda}{\lambda^2 B^2}\cos\alpha\sigma + \sin\alpha\sigma\right)e^{-i\lambda\tau}, \tag{2.28}$$

where c_λ is an arbitrary constant of integration and K_λ is given by

$$K_\lambda = \frac{\lambda^2 B^2 \sin\alpha\pi + \alpha\beta\sin\beta\pi}{\cos\beta\pi - \cos\alpha\pi}. \tag{2.29}$$

Let us study the solutions of Eq. (2.20) under condition (2.26). Eq. (2.20) is an equation in λ^2, so its solutions come in pairs $(\lambda,-\lambda)$. Solutions with $\lambda^2>0$ provide real λ and oscillating degrees of freedom. By contrast, solutions with $\lambda^2<0$ correspond to imaginary λ, for which the τ-exponentials are real.

For $\lambda^2>0$ and sufficiently large, the left-hand side of Eq. (2.20) can be expanded in powers of $x = m^2\kappa^2/\lambda^2 \ll 1$, with result

$$\begin{aligned}&\left(1+B^2\right)^2\sin^2\lambda\pi - x^2\left[\frac{\lambda^2\pi^2}{4}\left(1-B^2\right)^2\right.\\ &\quad\left. + \left(1+B^2\right)\left(\frac{\lambda\pi}{8}\sin 2\lambda\pi + \sin^2\lambda\pi\right)\right] + \mathcal{O}\left(x^3\right) = 0.\end{aligned} \tag{2.30}$$

The left-hand side is, up to order x^3, negative for integer λ and positive for non-integer λ. It follows that the left-hand side of Eq. (2.20), to which (2.30) is an approximation for large λ, must change its sign twice in the vicinity of every integer $n \gg |m\kappa|$, thus proving the existence of two solutions around n. These solutions can be found as power series in $m\kappa/n$ by making for λ in the neighborhood of n the ansatz

$$\lambda_n = n\sum_{k=0}^{\infty} a_k\left(\frac{m\kappa}{n}\right)^k, \qquad a_0 = 1,$$

where the coefficient a_0 has been taken equal to 1 since $\lambda = n$ solves Eq. (2.20) to lowest order. Substituting this ansatz in Eq. (2.20) and solving order by order in $m\kappa/n$, one obtains two different sets of solutions for the coefficients $\{a_k\}$, leading to

$$\lambda_n^{(1,2)} = n\left[1 \pm \frac{m^2\kappa^2}{2n^2}\,\frac{1-B^2}{1+B^2} + \mathcal{O}\left(\frac{m^4\kappa^4}{n^4}\right)\right].$$

This confirms the existence of two real eigenvalues for every large enough integer n, thus showing that there are infinitely many real solutions with $|\lambda| > |m\kappa|$.

By contrast, there is only a finite number of real solutions with $|\lambda| < |m\kappa|$ and this number depends on the value of $m\kappa$. This can be seen as follows. Assume, without loss of generality, that $m\kappa$ is in between two consecutive integers, so that $N \leqslant |m\kappa| < N+1$, with N a positive integer. Denote by N' the integer such that $N' < \sqrt{2}|m\kappa| \leqslant N'+1$. Study the sign of the right-hand side of Eq. (2.20) as a function of β by dividing the interval for β in subintervals $[0,1], [1,2], \ldots, [N', N'+1]$. It is not then very difficult to prove that

(i) for N' even there are $2(N'-N+1)$ real solutions, and

(ii) for N' odd the number of solutions is also $2(N'-N+1)$ if

$$\frac{2\sqrt{2}}{|m\kappa|\pi B^2}\sin\bigl(\sqrt{2}|m\kappa|\pi\bigr) + \cos\bigl(\sqrt{2}|m\kappa|\pi\bigr) + 1 > 0$$

and $2(N'-N)$ otherwise.

We come now to imaginary solutions. For $\lambda^2 < 0$, with $|\lambda| > m\kappa$, the left-hand side of Eq. (2.20) is positive definite and never vanishes. Hence imaginary solutions must have $|\lambda| < m\kappa$. Using similar arguments to those employed for real λ, it can be seen that in this case the number of solution for a given $m\kappa$ is $2(N+1)$, with N the integer such that $N < |m\kappa| \leqslant N+1$. We note that imaginary λ's occur due to the different signs with which $(X^1)^2$ and $(X^2)^2$ enter the background metric (1.2) and account for exponential growth of X^1 and X^2 at $\tau \to \pm\infty$. This is reminiscent of de Sitter space, for which space expands so fast that light rays cannot follow.

This analysis shows that there are infinitely many modes of type (2.27)–(2.28), of which a finite number of them have imaginary λ with $|\lambda| < |m\kappa|$, a finite number have real λ with $|\lambda| < |m\kappa|$, and infinitely many of them have real λ with $|\lambda| > |m\kappa|$. It is important to emphasize that this is so for arbitrary values of $m\kappa$, since Eq. (2.20) and condition (2.26) do not place any limitation on $m\kappa$. These modes can also be written in the following way, which will be very useful in some parts of this paper. The eigenvalue equation (2.20) can be recast as

$$F_+(\lambda)F_-(\lambda) = 0,$$

with $F_\pm(\lambda)$ functions given by

$$F_\pm(\lambda) = \frac{\alpha\beta}{\lambda^2 B^2} - \frac{(\cos\alpha\pi \pm 1)(\cos\beta\pi \mp 1)}{\sin\alpha\pi\,\sin\beta\pi}. \tag{2.31}$$

Condition (2.26) and the observation that $F_+(\lambda)$ and $F_-(\lambda)$ do not have common zeros imply that the set of solutions to the eigenvalue equation (2.20) is the union of the disjoint sets $\Lambda_+ = \{\lambda_+\}$ and $\Lambda_- = \{\lambda_-\}$ of solutions of the equations

$$F_\pm(\lambda_\pm) = 0. \tag{2.32}$$

It is then a matter of algebra to write X^1 and X^2 as

$$X^i_\lambda(\tau,\sigma) = \begin{cases} X^i_+(\tau,\sigma) & \text{if } \lambda \in \Lambda_+, \\ X^i_-(\tau,\sigma) & \text{if } \lambda \in \Lambda_-, \end{cases} \qquad i = 1,2, \tag{2.33}$$

with $X^i_\pm$ given by

$$X^1_\pm(\tau,\sigma) = ic_\lambda \frac{\alpha}{\lambda B}\left(\cos\beta\sigma + \frac{\sin\beta\pi}{\cos\beta\pi \mp 1}\sin\beta\sigma\right)e^{-i\lambda\tau}, \tag{2.34}$$

$$X^2_\pm(\tau,\sigma) = -c_\lambda\left(\frac{\cos\alpha\pi \pm 1}{\sin\alpha\pi}\cos\alpha\sigma + \sin\alpha\sigma\right)e^{-i\lambda\tau}. \tag{2.35}$$

Putting all cases together, we conclude that the solution for the boundary problem for X^1, X^2 is:

(1) If $m\kappa$ is not an integer and its square is not an even integer, the only modes that occur are those in Eqs. (2.34)–(2.35), corresponding to $\lambda \in \Lambda_\pm$.

(2) If $m\kappa$ is not an integer but its square is an even integer, one has in addition the modes (k,l) in (2.22)–(2.23).

(3) If $m\kappa$ is an even integer, there is one additional mode, $X^1_{\rm e}, X^2_{\rm e}$ in (2.18)–(2.19).

(4) Finally, if $m\kappa$ is an odd integer, the only modes that occur are those in (1) and $X^1_{\rm o}, X^2_{\rm o}$ in (2.16)–(2.17).

We summarize all these situations by writing

$$X^i(\tau,\sigma) = \sum_{\lambda\in\Lambda_\pm} X^i_\lambda + \delta_{m^2\kappa^2,\rm even} \sum_{(k,l)} X^i_{(k,l)} + \delta_{m\kappa,\rm even} X^i_{\rm e} + \delta_{m\kappa,\rm odd} X^i_{\rm o}. \tag{2.36}$$

The mode expansions for the momenta P_i, P_a follow from their expressions (2.11)–(2.12) in terms of string coordinates and the mode expansions for the string coordinates. For the flat a-directions it is trivial to arrive at

$$2\pi\alpha' P_a = d^a_0 + \sum_{n\neq 0}^{\infty} c^a_n \cos n\sigma\, e^{-in\tau}.$$

For the i-directions we have

$$P_i(\tau,\sigma) = \sum_{\lambda\in\Lambda_\pm} P_{i,\lambda} + \delta_{m^2\kappa^2,\rm even} \sum_{(k,l)} P_{i,(k,l)} + \delta_{m\kappa,\rm even} P_{i,\rm e} + \delta_{m\kappa,\rm odd} P_{i,\rm o}, \tag{2.37}$$

where the explicit expressions for the various contributions to the right-hand side can be found in Appendix A.

2.4. The string center of mass coordinates and the string total momentum

The string center of mass coordinates

$$x^{i,a}_{\rm cm}(\tau) = \frac{1}{\pi}\int_0^\pi d\sigma\, X^{i,a}(\tau,\sigma)$$

and the string total momentum

$$p_{i,a}(\tau) = \int_0^\pi d\sigma\, P_{i,a}(\tau,\sigma)$$

are straightforward to calculate from the mode expansions in the previous subsection. Let us consider for instance the total momentum. For the flat a-directions integration over $d\sigma$ gives the standard result $p_a = d^a_0/2\alpha'$. The a-component of the total string momentum is thus given by one of the string modes in that direction. The situation for the 1 and 2-component is very different. Indeed, integration over $d\sigma$ of the equations in Appendix A yields

$$p_1(\tau)=\frac{\delta_{m\kappa,\mathrm{even}}}{\pi\alpha'}\frac{B^2}{m\kappa}b_{\mathrm{e}}\sinh\left(\frac{mk\pi}{2}\right)$$
$$-\frac{m^2\kappa^2}{\pi\alpha'}\left[\delta_{m^2\kappa^2,\mathrm{even}}\sum_{\substack{(k,l)\\ k\ \mathrm{odd}}}\frac{b_{\lambda(k,l)}}{\beta\lambda^2}e^{-i\lambda\tau}+\sum_{\lambda\in\Lambda_-}\frac{Bc_\lambda}{\beta^2}\frac{\cos\alpha\pi-1}{\sin\alpha\pi}e^{-i\lambda\tau}\right] \tag{2.38}$$

and

$$p_2(\tau)=\frac{\delta_{m\kappa,\mathrm{odd}}}{\pi\alpha'}\left[Bb_{\mathrm{o}}\tau\sinh\left(\frac{mk\pi}{2}\right)-a_{\mathrm{o}}B\right]$$
$$+\frac{im^2\kappa^2}{\pi\alpha'}\left[\delta_{m^2\kappa^2,\mathrm{even}}\sum_{\substack{(k,l)\\ k\ \mathrm{odd}}}\frac{\lambda a_{\lambda(k,l)}}{\alpha}e^{-i\lambda\tau}+\sum_{\lambda\in\Lambda_+}\frac{c_\lambda}{\lambda\alpha}e^{-i\lambda\tau}\right]. \tag{2.39}$$

The components p_1 and p_2 receive contributions from all the string modes in those directions. More importantly, p_1 and p_2 are not conserved since their derivatives with respect to τ do not vanish. This is not a surprise, for the plane-wave metric (1.2) is not invariant under translations in the 1 and 2-directions. Upon quantization, we therefore do not expect the eigenvalues of the corresponding operators to play a significant rôle. It is trivial to convince oneself that this collective nature of p_i is also true for the string center of mass coordinates, whose explicit expression can be trivially obtained through integration over $d\sigma$.

2.5. Case $|m\kappa|\ll 1$

We finish by considering the regime $|m\kappa|\ll 1$. Since $m\kappa$ is not an integer, nor $m^2\kappa^2$ is an even integer, the only modes that exist in this case are those in (2.27)–(2.28), or equivalently (2.34)–(2.35). Furthermore, the mode eigenvalues λ can be explicitly found as formal power series in $m\kappa$ by making the ansatz $\lambda=\sum_0^\infty b_k(m\kappa)^k$ and solving Eq. (2.20) for the coefficients b_k order by order. Proceeding in this way we obtain:

(i) *Imaginary eigenvalues*. As already mentioned, they have $|\lambda|<|m\kappa|$. The algebra shows that there are only two of them, $\Lambda^{\mathrm{I}}=\{\pm i\lambda^{\mathrm{I}}\}$, given by

$$\lambda^{\mathrm{I}}=\frac{m\kappa}{\sqrt{1+B^2}}\left[1+\frac{(m\kappa)^2}{12}\frac{\pi^2B^2}{1+B^2}+\frac{(m\kappa)^4}{1440}\frac{\pi^4B^2(5B^2-24)}{(1+B^2)^2}+\mathcal{O}\big(m^6\kappa^6\big)\right]. \tag{2.40}$$

In terms of Eqs. (2.32), they happen to solve $F_-(\lambda)=0$, thus belong to Λ_-.

(ii) *Real eigenvalues with* $|\lambda|<|m\kappa|$. There are also two of them, $\Lambda^{\mathrm{R}}=\{\pm\lambda^{\mathrm{R}}\}$, where

$$\lambda^{\mathrm{R}}=\frac{m\kappa}{\sqrt{1+B^2}}\left[1-\frac{(m\kappa)^2}{12}\frac{\pi^2B^2}{1+B^2}+\frac{(m\kappa)^4}{1440}\frac{\pi^4B^2(5B^2-24)}{(1+B^2)^2}+\mathcal{O}\big(m^6\kappa^6\big)\right]. \tag{2.41}$$

They are now solutions of $F_+(\lambda)=0$, thus are in Λ_+.

(iii) *Real eigenvalues with* $|\lambda|>|m\kappa|$. They read

$$\left\{\begin{matrix}\lambda_n\\ \tilde\lambda_n\end{matrix}\right\}=n\left[1\pm\frac{m^2\kappa^2}{2n^2}\frac{1-B^2}{1+B^2}-\frac{m^4\kappa^4}{8n^4}\frac{B^4-6B^2+1}{(1+B^2)^2}+\mathcal{O}\big(m^6\kappa^6\big)\right], \tag{2.42}$$

where n is a non-zero integer and the $+/-$ signs on the right-hand side correspond to $\lambda_n/\tilde{\lambda}_n$ on the left side. We will use the notation $\Lambda := \{\lambda_n\}$ and $\tilde{\Lambda} := \{\tilde{\lambda}_n\}$. These eigenvalues can be reorganized in terms of solutions of Eqs. (2.32) as

$$\lambda_+^{(n)} = \begin{cases} \lambda_n & \text{if } n \text{ even,} \\ \tilde{\lambda}_n & \text{if } n \text{ odd,} \end{cases} \qquad \lambda_-^{(n)} = \begin{cases} \tilde{\lambda}_n & \text{if } n \text{ even,} \\ \lambda_n & \text{if } n \text{ odd.} \end{cases} \tag{2.43}$$

It is instructive to compare the mode eigenvalues with those for the open string in flat space–time and zero antisymmetric field, i.e., with $m = B = 0$. In that case, X^1 and X^2 have the same expansion as in (2.14) and the mode eigenvalues are the integers. The flat zero mode $\lambda_{\text{flat}} = 0$ has multiplicity four in the 1, 2-directions, for there are four arbitrary constants of integration, which in our notation would be denoted c_0^1, c_0^2, d_0^1, d_0^2. Every pair $(n, -n)$ of non-zero flat modes is also 4-degenerate in these directions, for in each direction there are two complex coefficients c_{-n}^i and c_n^i and one complex constraint $(c^i)_n^\star = c_{-n}^i$. If m and B are switched on, the flat zero mode unfolds into two non-zero imaginary modes $(i\lambda^{\mathrm{I}}, -i\lambda^{\mathrm{I}})$ and two non-zero real modes $(\lambda^{\mathrm{R}}, -\lambda^{\mathrm{R}})$, and every pair of flat modes $(n, -n)$ unfolds into four modes $(\lambda_n, \tilde{\lambda}_n, \lambda_{-n}, \tilde{\lambda}_{-n})$. Whereas in Minkowski space–time, the string center of mass and string total momentum are independent degrees of freedom associated to the 4-degenerate zero mode, in our plane-wave background they are collective quantities.

3. Quantization

There is a discussion in the literature for $m = 0$ as for how to quantize the open string with non-trivial boundary conditions like those in (2.7) and (2.8). It seems to be a widespread believe that these boundary conditions impeach the use of canonical quantization. In fact, for $m = 0$, Dirac quantization, with the boundary conditions regarded as constraints, has been used as an alternative. The problem that arises then is whether the boundary conditions should be regarded as first or second class, and this is not a trivial choice for they lead to different results [26,27].

We will use plain canonical quantization and show that there is nothing wrong with it. Our approach consists of two steps. In the first one we compute the symplectic form in terms of the modes. This is straightforward, since the action is first order in time derivatives and it is well known how to proceed in these cases [28,29]. The resulting symplectic form will be non-singular, so it has an inverse. Its inverse defines, upon standard canonical quantization [28], the commutation relations for the quantum theory. We emphasize that the calculation of the symplectic form may be involved but, as pointed out in Refs. [28,29], as far as it is non-singular there is nothing wrong with canonical quantization and there is no need to introduce constraints of any type. It is also worth noting in this respect that the boundary conditions have already been taken into account in solving the classical equations of motions, so one would naïvely expect the symplectic form to already account for them. We will see that this quantization method is consistent with the equal-time commutation relations

$$\left[X^i(\tau, \sigma), P_j(\tau, \sigma')\right] = i\delta^i_j \delta(\sigma - \sigma'). \tag{3.1}$$

In Section 5 we will explicitly construct the Fock–Hilbert space for the theory and find the Hamiltonian spectrum.

3.1. Symplectic form and canonical quantization

The symplectic form

$$\Omega = \int_0^\pi d\sigma \left(\mathbf{d}P_i \wedge \mathbf{d}X^i + \mathbf{d}P_a \wedge \mathbf{d}X^a\right)$$

is the sum of two contributions, which we will call Ω_{pp} and $\Omega_{\rm flat}$. They respectively arise from the modes in the i-directions and the flat a-directions. Since they do not mix, the symplectic form can be studied by separately looking at each one of these two sectors.

Let us first look at $\Omega_{\rm flat}$. Recalling the mode expansions for X^a and P_a, one easily arrives at

$$\Omega_{\rm flat} = \int_0^\pi d\sigma\, \mathbf{d}P_a \wedge \mathbf{d}X^a = \frac{1}{2\alpha'} \sum_{a=3}^{D-2} \left(\mathbf{d}d_0^a \wedge \mathbf{d}c_0^a - \sum_{n\neq 0} \frac{i}{2n} \mathbf{d}c_n^a \wedge \mathbf{d}c_{-n}^a\right). \tag{3.2}$$

This can be written as

$$\Omega_{\rm flat} = \frac{1}{2} \Omega_{MM'}\, \mathbf{d}A_M \wedge \mathbf{d}A_{M'}, \tag{3.3}$$

where $\{A_M\} = \{d_0^a, c_0^a, c_n^a\}$ and a summation over indices $M = (a, n)$ and $M' = (a', n')$ is understood. The form $\Omega_{\rm flat}$ is non-singular and can be inverted. Upon quantization, the amplitudes $\{A_M\}$ become operators with commutation relations given by the inverse of Ω as

$$[A_M, A_{M'}] = i\left(\Omega^{-1}\right)_{MM'}. \tag{3.4}$$

This yields the standard commutation relations

$$\left[c_0^a, d_0^b\right] = 2i\alpha'\delta^{ab}, \qquad \left[c_n^a, c_m^b\right] = 2\alpha' n \delta^{ab} \delta_{n+m,0}.$$

Reality of the field operators X^a imply that c_0^a and d_0^a are Hermitean and that $c_{-n}^a = (c_n^a)^\dagger$. So far, this is the same analysis as for Minkowski space–time.

To compute Ω_{pp} it is most convenient to use Eq. (2.11) and write P_i in terms of derivatives of X^i with respect to τ and σ. This gives

$$\Omega_{pp} = \int_0^\pi d\sigma\, \mathbf{d}P_i \wedge \mathbf{d}X^i = \tilde{\Omega}_{pp} + \bar{\Omega}_{pp},$$

where $\tilde{\Omega}_{pp}$ and $\bar{\Omega}_{pp}$ read

$$\tilde{\Omega}_{pp} = \frac{1}{2\pi\alpha'} \int_0^\pi d\sigma\, \mathbf{d}\left(\partial_\tau X^i\right) \wedge \mathbf{d}X^i \tag{3.5}$$

and

$$\bar{\Omega}_{pp} = \frac{B}{2\pi\alpha'} \mathbf{d}X^1 \wedge \mathbf{d}X^2 \Big|_{\sigma=0}^{\sigma=\pi}. \tag{3.6}$$

As compared to the flat a-directions, for which $2\pi\alpha' P_a = \partial_\tau X^a$, the non-trivial boundary conditions not only modify the modes but also add a boundary term $\bar{\Omega}_{pp}$ to the symplectic form.

Computation of the boundary piece $\bar{\Omega}_{pp}$ is straightforward. To calculate $\tilde{\Omega}_{pp}$, we use the mode expansion (2.36), integrate over $d\sigma$, rearrange the mode sums and employ that the eigenvalues $\lambda \in \Lambda_\pm$ are solutions of Eqs. (2.32). After some very long, but also very straightforward algebra, we obtain that

$$\Omega_{pp} = \Omega_{\Lambda_\pm} + \delta_{m^2\kappa^2,\mathrm{even}}\Omega_{\{(k,l)\}} + \delta_{m\kappa,\mathrm{even}}\Omega_{\mathrm{e}} + \delta_{m\kappa,\mathrm{odd}}\Omega_{\mathrm{o}}. \tag{3.7}$$

The various contributions in this equation are given by

$$\Omega_{\Lambda_\pm} = \frac{i}{2\pi\alpha'} \sum_{\lambda\in\Lambda_\pm} f(\lambda)\, \mathbf{d}c_\lambda \wedge \mathbf{d}c_{-\lambda}, \tag{3.8}$$

$$\Omega_{\{(k,l)\}} = -\frac{i}{4\alpha' B} \sum_{(k,l)} \big[f_a(\lambda)\, \mathbf{d}a_{\lambda(k,l)} \wedge \mathbf{d}a_{-\lambda(k,l)} + f_b(\lambda)\mathbf{d}b_{\lambda(k,l)} \wedge \mathbf{d}b_{-\lambda(k,l)} \big], \tag{3.9}$$

$$\Omega_{\mathrm{e}} = -\frac{1}{4\alpha'} \cosh\left(\frac{m\kappa\pi}{2}\right) \mathbf{d}a_{\mathrm{e}} \wedge \mathbf{d}b_{\mathrm{e}}, \tag{3.10}$$

$$\Omega_{\mathrm{o}} = -\frac{1}{4\alpha'} \sinh\left(\frac{m\kappa\pi}{2}\right) \mathbf{d}a_{\mathrm{o}} \wedge \mathbf{d}b_{\mathrm{o}}, \tag{3.11}$$

where $f(\lambda)$, $f_a(\lambda)$ and $f_b(\lambda)$ read

$$f(\lambda) = -\frac{\lambda\alpha(\cos\alpha\pi \pm 1)}{\sin\alpha\pi} \left[\frac{2(m\kappa)^4}{\lambda^2\alpha^2\beta^2} \pm \frac{\pi}{\alpha\sin\alpha\pi} \mp \frac{\pi}{\beta\sin\beta\pi} \right], \tag{3.12}$$

$$f_a(\lambda) = \frac{\lambda}{B^2}\left(1 + B^2 - \frac{m^2\kappa^2}{\lambda^2}\right), \tag{3.13}$$

$$f_b(\lambda) = \frac{1}{\lambda B^2}\left(1 + B^2 + \frac{m^2\kappa^2}{\lambda^2}\right). \tag{3.14}$$

In accordance with the notation that we are using, the double signs $\pm$ on the right of the equation for $f(\lambda)$ apply, respectively, to the eigenvalues $\lambda_\pm$ solving Eqs. (2.32).

We make at this point two comments concerning the computation of Ω_{pp}. The first one is that the only non-zero components $\Omega_{MM'}$ of the symplectic form have $M + M' = 0$, where M labels all the existing mode $\{A_M\} = \{a_{\mathrm{o}}, b_{\mathrm{o}}, a_{\mathrm{e}}, b_{\mathrm{e}}, a_{\lambda(k,l)}, b_{\lambda(k,l)}, c_\lambda\}$. Some authors call this orthogonality of modes. Note in particular that there is not any mixing of the modes for $m^2\kappa^2 = \mathrm{even}$, $m\kappa = \mathrm{even}$ and $m\kappa = \mathrm{odd}$ among themselves, nor with modes $\lambda \in \Lambda_\pm$. The second comment is to emphasize that the result above for Ω_{pp} follows straightforwardly from Eqs. (3.5)–(3.6) after plain integration over $d\sigma$, without any assumption whatsoever.

The form Ω_{pp} is non-singular and has an inverse Ω_{pp}^{-1}. Canonical quantization is then straightforward. The amplitudes $\{A_M\}$ become operators. Hermiticity of X^i implies that a_{o}, b_{o}, a_{e}, b_{e} and c_λ ($\lambda \in \Lambda_\pm$ imaginary) are Hermitean and that

$$a^\dagger_{\lambda(k,l)} = a_{-\lambda(k,l)}, \qquad b^\dagger_{\lambda(k,l)} = b_{-\lambda(k,l)}, \qquad c^\dagger_\lambda = c_{-\lambda} (\lambda \in \Lambda_\pm \mathrm{real}).$$

The commutation rules are obtained from the inverse of Ω_{pp} as in (3.3)–(3.4), the only non-trivial commutation relations being

$$\big[c_\lambda, c^\dagger_\lambda\big] = -\frac{\pi\alpha'}{f(\lambda)}, \tag{3.15}$$

$$\big[a_{\lambda(k,l)}, a^{\dagger}_{\lambda(k,l)}\big] = \frac{2\alpha'}{f_a(\lambda)}, \qquad \big[b_{\lambda(k,l)}, b^{\dagger}_{\lambda(k,l)}\big] = \frac{2\alpha'}{f_b(\lambda)}, \tag{3.16}$$

$$[a_{\rm e}, b_{\rm e}] = -4i\alpha' \operatorname{cosech}\left(\frac{m\kappa\pi}{2}\right), \tag{3.17}$$

$$[a_{\rm o}, b_{\rm o}] = -4i\alpha' \operatorname{sech}\left(\frac{m\kappa\pi}{2}\right). \tag{3.18}$$

We note that $f(\lambda)$ is real for λ real and imaginary for λ imaginary. The space of states on which these operators act and their action is given in Section 5. Let us move on to study the consistency of this quantization with the canonical commutation relations (3.1).

3.2. Canonical commutation relations

The commutator $[X^i(\tau,\sigma), P_j(\tau,\sigma')]$ can be computed by replacing X^i and P_j with their mode expansions and using the relations (3.15)–(3.18) for the mode operators in them. In doing so, the τ-dependence of the commutator is removed and a mode sum is left. This sum involves in particular an infinite sum over mode eigenvalues $\lambda \in \Lambda_\pm$ whose terms are products of sines and cosines at $\alpha\sigma$, $\beta\sigma$, $\alpha\sigma'$, $\beta\sigma'$ with complicated coefficients involving the function $f(\lambda)$. We do not see a way to perform this sum in closed form and obtain a compact expression for the commutator. We will instead expand the commutator in powers of $m\kappa$ and perform the mode sums order by order in $m\kappa$. We do this in the sequel.

If $|m\kappa| \ll 1$, the only modes that exist are those in Eqs. (2.34)–(2.35). We recall from Section 2.4 that in this case the mode eigenvalues are given by $\Lambda_{\rm I} = \{\pm i\lambda^{\rm I}\}$, $\Lambda_{\rm R} = \{\pm\lambda^{\rm R}\}$, $\Lambda = \{\lambda_n\}$ and $\tilde\Lambda = \{\tilde\lambda_n\}$ in Eqs. (2.40)–(2.42), with $n = \pm1, \pm2, \ldots$. We denote by $\{c^{\rm I}_\pm\}$, $\{c^{\rm R}_\pm\}$, $\{c_n\}$ and $\{\tilde c_n\}$ the corresponding annihilation and creation operators, for which hermiticity of the string position operators implies

$$\big(c^{\rm I}_\pm\big)^\dagger = c^{\rm I}_\pm, \qquad \big(c^{\rm R}_+\big)^\dagger = c^{\rm R}_-, \qquad (c_n)^\dagger = c_{-n}, \qquad (\tilde c_n)^\dagger = \tilde c_{-n}.$$

Expanding the right-hand side of Eq. (3.15) in powers of $m\kappa$, we obtain the following commutations relations for them:

$$\big[c^{\rm I}_+, c^{\rm I}_-\big] = -\frac{i\alpha' B^2}{2(2+B^2)(1+B^2)^{1/2}} \frac{1}{m\kappa}\left[1 + \frac{\pi^2 B^2 (m\kappa)^2}{6(2+B^2)} + \mathcal{O}\big(m^4\kappa^4\big)\right], \tag{3.19}$$

$$\big[c^{\rm R}_+, c^{\rm R}_-\big] = -\frac{\alpha'\pi^2 B^2}{8(1+B^2)^{3/2}} m\kappa\left[1 + \frac{\pi^2(1-B^2)(m\kappa)^2}{6(1+B^2)} + \mathcal{O}\big(m^4\kappa^4\big)\right], \tag{3.20}$$

$$[c_n, c_k] = \frac{\alpha'\pi^2 B^4}{4n^3(1+B^2)^3}(m\kappa)^4\left[1 - \frac{(3-5B^2)(m\kappa)^2}{2n^2(1+B^2)} + \mathcal{O}\big(m^4\kappa^4\big)\right]\delta_{n+k,0}, \tag{3.21}$$

$$[\tilde c_n, \tilde c_k] = \frac{\alpha' B^2}{n(1+B^2)}\left[1 + \frac{3(m\kappa)^2}{2n^2(1+B^2)} + \mathcal{O}\big(m^4\kappa^4\big)\right]\delta_{n+k,0}, \tag{3.22}$$

all other commutators being zero. The commutator $[X^i(\tau,\sigma), P_j(\tau,\sigma')]$ can then be written as a sum

$$\big[X^i(\tau,\sigma), P_j(\tau,\sigma')\big] = \sum_{\omega=I,R,n,\tilde n} C^i{}_j(\omega;\sigma,\sigma')$$

of four contributions ${C^i}_j(\omega;\sigma,\sigma')$ arising from the four sets in which the modes have been organized. Each one of these contributions is a power series in $m\kappa$, depends on σ and σ' and can be computed with relative ease order by order. To illustrate this, let us take as an example $i=j=2$. After some algebra we obtain

$$ {C^2}_2(\mathrm{I};\sigma,\sigma') = -\frac{i\alpha' B^2(m\kappa)^2}{2(1+B^2)}\left(\frac{\pi^2}{2}-\pi\sigma-\pi\sigma'+2\sigma\sigma'\right)+\mathcal{O}\big(m^4\kappa^4\big), \tag{3.23} $$

$$ {C^2}_2(\mathrm{R};\sigma,\sigma') = i\alpha' -\frac{i\alpha' B^2(m\kappa)^2}{2(1+B^2)}\left(\frac{\pi^2}{6}+\pi\sigma-\sigma^2-\pi\sigma'+\sigma'^2\right)+\mathcal{O}\big(m^4\kappa^4\big), \tag{3.24} $$

$$ \begin{aligned} {C^2}_2(\Lambda;\sigma,\sigma') = 2i\alpha'\sum_{n=1}^{\infty}\cos n\sigma'\bigg[\cos n\sigma + \frac{B^2(m\kappa)^2}{1+B^2}\bigg(\frac{\cos n\sigma}{n^2}+\frac{\sigma\sin n\sigma}{n} \\ -\frac{\pi}{2}\frac{\sin n\sigma}{n}\bigg)\bigg]+\mathcal{O}\big(m^4\kappa^4\big), \end{aligned} \tag{3.25} $$

$$ {C^2}_2(\tilde{\Lambda};\sigma,\sigma') = -\frac{i\alpha' B^2(m\kappa)^2}{1+B^2}\sum_{n=1}^{\infty}\big(2\sigma'-\pi\big)\frac{\sin n\sigma\cos n\sigma'}{n}+\mathcal{O}\big(m^4\kappa^4\big). \tag{3.26} $$

It follows from inspection of these formuli that only ${C^2}_2(\mathrm{R})$ and ${C^2}_2(\Lambda)$ carry contributions of order zero in $m\kappa$. These are easily summed by recalling that, for functions defined on $[0,\pi]$ with vanishing derivatives at the boundary, Dirac's delta function has the representation

$$ \pi\delta(\sigma-\sigma') = 1+2\sum_{n=1}^{\infty}\cos n\sigma\cos n\sigma'. $$

Hence

$$ \big[X^2(\tau,\sigma),\,P_2(\tau,\sigma')\big]_0 = i\alpha'\delta(\sigma-\sigma'), $$

where the subscript 0 refers to the order in $m\kappa$. To sum the order-two in $m\kappa$ contributions, it is convenient to introduce variables $\sigma_\pm=\sigma\pm\sigma'$, which take values $\sigma_-\in[-\pi,\pi]$ and $\sigma_+\in[0,2\pi]$. In terms of these, we have

$$ \big[{C^2}_2(\Lambda)+{C^2}_2(\tilde{\Lambda})\big]_2 = \frac{i\alpha' B^2(m\kappa)^2}{2(1+B^2)}\big[F_2(\sigma_-)+F_2(\sigma_+)+\sigma_-F_1(\sigma_-)+\sigma_-F_1(\sigma_+)\big], $$

where F_1 and F_2 stand for the Fourier series

$$ F_1(\sigma_-) := 2\sum_{n=1}^{\infty}\frac{\sin n\sigma_-}{n} = \begin{cases} \frac{\pi|\sigma_-|}{\sigma_-}-\sigma_- & \text{if } 0<|\sigma_-|<\pi, \\ 0 & \text{if } \sigma_-=0,\pm\pi, \end{cases} \tag{3.27} $$

$$ F_1(\sigma_+) := 2\sum_{n=1}^{\infty}\frac{\sin n\sigma_+}{n} = \begin{cases} \pi-\sigma_+ & \text{if } 0<\sigma_+<2\pi, \\ 0 & \text{if } \sigma_+=0,2\pi, \end{cases} \tag{3.28} $$

$$ F_2(\sigma_-) := 2\sum_{n=1}^{\infty}\frac{\cos n\sigma_-}{n^2} = \frac{\sigma_-^2}{2}-\pi|\sigma_-|+\frac{\pi^2}{3}, \tag{3.29} $$

$$ F_2(\sigma_+) := 2\sum_{n=1}^{\infty}\frac{\cos n\sigma_+}{n^2} = \frac{\sigma_+^2}{2}-\pi\sigma_++\frac{\pi^2}{3}. \tag{3.30} $$

Putting together all contributions of order two in Eqs. (3.23)–(3.26), we obtain

$$\big[X^2(\tau,\sigma),\,P_2(\tau,\sigma')\big]_2 = 0,$$

in agreement with (3.1). Proceeding in the same way, it is straightforward to see that the commutation relations in (3.1) also hold for other values of i and j, so we can write

$$\big[X^i(\tau,\sigma),\,P_k(\tau,\sigma')\big] = i\alpha'\delta^i{}_j\delta(\sigma-\sigma') + \mathcal{O}\big(m^4\kappa^4\big).$$

This proves the consistency of the quantization procedure used here with equal-time canonical commutation relations, at least up to order $m^4\kappa^4$.

We find quite surprising the asymmetric rôle that each type of mode plays in this analysis, yet all combine to produce the desired result. It is also worth noting that $C^i{}_j(\Lambda)$ and $C^i{}_j(\tilde{\Lambda})$ will involve to any order in $m\kappa$ polynomials in $\sigma_\pm$ multiplied with convergent Fourier series of $\sigma_\pm$, thus becoming a question of algebra force to go to higher orders in $m\kappa$. It is by now clear that canonical quantization works and that it does because the symplectic form is non-singular.

4. Non-commutative wave fronts

The plane-wave metric (1.2) foliates space–time by null surfaces $X^+ = \text{const}$. We show next that these spaces are non-commutative. The commutator $[X^1(\tau,\sigma), X^2(\tau,\sigma')]$ can be computed by replacing X^1 and X^2 with their mode expansions and using the commutation relations (3.15)–(3.18) for the mode operators. This results in

$$\begin{aligned}\big[X^1(\tau,\sigma),\,X^2(\tau,\sigma')\big] = i\big[&\Theta_{\Lambda_\pm}(\sigma,\sigma') + \delta_{m^2\kappa^2,\mathrm{even}}\Theta_{\{(k,l)\}}(\sigma,\sigma')\\ &+ \delta_{m\kappa,\mathrm{even}}\Theta_{\mathrm{e}}(\sigma,\sigma') + \delta_{m\kappa,\mathrm{odd}}\Theta_{\mathrm{o}}(\sigma,\sigma')\big],\end{aligned} \tag{4.1}$$

where the contribution $\Theta_{\Lambda_\pm}(\sigma,\sigma')$ is given by

$$\begin{aligned}\Theta_{\Lambda_\pm}(\sigma,\sigma') = \frac{1}{2B}\sum_{\lambda\in\Lambda_\pm}\frac{\alpha}{\lambda f(\lambda)}&\left(\cos\beta\sigma + \frac{\sin\beta\pi}{\cos\beta\pi \mp 1}\sin\beta\sigma\right)\\ \times&\left(\frac{\cos\alpha\pi \pm 1}{\sin\alpha\pi}\cos\alpha\sigma' + \sin\alpha\sigma'\right)\end{aligned} \tag{4.2}$$

and $\Theta_{\{(k,l)\}}(\sigma,\sigma')$, $\Theta_{\mathrm{e}}(\sigma,\sigma')$ and $\Theta_{\mathrm{o}}(\sigma,\sigma')$ read

$$\Theta_{\{(k,l)\}}(\sigma,\sigma') = -4\alpha' B\sum_{(k,l)}\left[\frac{\alpha\cos\beta\sigma\,\sin\alpha\sigma'}{\lambda^2(1+B^2)-m^2\kappa^2} + \frac{\beta\sin\beta\sigma\,\cos\alpha\sigma'}{\lambda^2(1+B^2)+m^2\kappa^2}\right], \tag{4.3}$$

$$\Theta_{\mathrm{e}}(\sigma,\sigma') = \frac{4\alpha' B}{m\kappa}\,\mathrm{cosech}\left(\frac{m\kappa\pi}{2}\right)\cos(m\kappa\sigma)\sinh\left[m\kappa\left(\frac{\pi}{2}-\sigma'\right)\right], \tag{4.4}$$

$$\Theta_{\mathrm{o}}(\sigma,\sigma') = \frac{4\alpha' B}{m\kappa}\,\mathrm{sech}\left(\frac{m\kappa\pi}{2}\right)\cos(m\kappa\sigma)\cosh\left[m\kappa\left(\frac{\pi}{2}-\sigma'\right)\right]. \tag{4.5}$$

We recall that the sum in $\Theta_{\{(k,l)\}}(\sigma,\sigma')$ is over the finite number of solutions (k,l) of Eq. (2.25) and that α and β in this sum are as in (2.24), so the contributions (4.3)–(4.5) do not pose any problems.

The most complicated piece to understand is the contribution $\Theta_{\Lambda_\pm}(\sigma,\sigma')$. We may proceed as in Section 3 and consider $|m\kappa| \ll 1$. In this case only $\Theta_{\Lambda_\pm}(\sigma,\sigma')$ contributes to the commutator $[X^1, X^2]$. Expanding the right-hand side of Eq. (4.2) in powers of $m\kappa$, the sum over modes can then be performed order by order in $m\kappa$, so that $\Theta(\sigma,\sigma')$ becomes a power series

$$\Theta(\sigma,\sigma')=\sum_{k=0}^{\infty}\Theta_{2k}(\sigma,\sigma')(m\kappa)^{2k}$$

whose coefficients are explicit functions of σ and σ'. The first two terms of this series are calculated in Appendix B. We exhibit here the result. At the string endpoints we obtain

$$\Theta(0,0)=-\Theta(\pi,\pi)=\frac{\alpha'\pi B}{1+B^2}\left[1+\frac{\pi^2(m\kappa)^2}{6(1+B^2)}+\mathcal{O}\big(m^4\kappa^4\big)\right], \tag{4.6}$$

whereas at $\sigma+\sigma'\neq 0, 2\pi$ we have

$$\begin{aligned}\Theta(\sigma,\sigma')=\frac{\alpha' B(m\kappa)^2}{(1+B^2)^2}\bigg\{&B^2\Big[-\frac{\sigma}{6}\big(\sigma^2-3\sigma'^2\big)+\frac{\pi}{4}\big(\sigma^2-\sigma'^2-2\sigma\sigma'\big)-\frac{\pi}{12}(\sigma-3\sigma')\Big]\\&-\frac{\sigma}{12}\big(7\sigma^2+9\sigma'^2\big)+\frac{\pi}{8}\big(7\sigma^2+3\sigma'^2+6\sigma\sigma'\big)-\frac{\pi^2}{4}(3\sigma+\sigma')+\frac{\pi^3}{6}\\&+\frac{\pi}{8}|\sigma-\sigma'|\big[2B^2\big(\sigma^2+\sigma'-\pi\big)+5\sigma-\sigma'-2\pi\big]\bigg\}+\mathcal{O}\big(m^4\kappa^4\big).\end{aligned} \tag{4.7}$$

The limit $m\to 0$ is smooth and reproduces the results in the literature. In fact, as $m\to 0$, that is, as Minkowski space–time is approached, only the first term in (4.6) survives and the results in Ref. [19] are recovered. For $m\neq 0$, two novelties are found: non-commutativity at the string endpoints receives m-dependent corrections, and non-commutativity occurs for arbitrary values of σ and σ', so that it extends all along the string. Even for $\sigma=\sigma'\neq 0,\pi$ non-commutativity pervades, since in that case

$$\Theta(\sigma,\sigma)=\frac{\alpha' B(m\kappa)^2}{6(1+B^2)^2}(2\sigma-\pi)\big[B^2\sigma(\sigma-\pi)-(2\sigma-\pi)^2\big]+\mathcal{O}\big(m^4\kappa^4\big)\neq 0.$$

At the string midpoint $\sigma=\sigma'=\pi/2$ one has $\Theta=0$, not only for $m^2\kappa^2\ll 1$ but also for arbitrary $m\kappa$ since the right-hand side of (4.2) vanishes. Note also that commutativity is recovered as $B\to 0$.

The results in this section may be viewed from two perspectives. The first one is to assume a constant background field $B_{12}=B$ and that the string endpoints move freely, except for the boundary conditions imposed by the presence of the B field. The endpoints are then not distinguished by non-commutativity. The second one is to assume that B_{ij} vanishes but that the string endpoints are constrained to move on a D1-brane located at x_0^a on which a constant magnetic field $F_{12}=B$ is defined. The boundary conditions for X^1 and X^2 then remain unchanged while those for X^a become $X^a|_{\sigma=0,\pi}=x_0^a$. The only difference with the situation discussed here is that the mode expansion for X^a is no longer (2.14) but rather

$$X^a(\tau,\sigma)=x_0^a+\sum_{n\neq 0}i\frac{c_a^a}{n}\sin n\sigma\, e^{-in\sigma}.$$

This only introduces some trivial modifications in the analysis of the flat a-directions [19]. From this point of view, the plane-wave metric extends non-commutativity outside the D1-brane.

5. The Fock–Hilbert space and the spectrum

We want to solve the eigenvalue problem

$$H|\psi\rangle = E|\psi\rangle,$$

where the Hamiltonian is the integral over σ of the Hamiltonian density $\mathcal{H}$ in Eq. (2.13). As discussed in Sections 2.1 and 2.3, the classical Hamiltonian is not positive. This translates, upon quantization, into an unbounded Hamiltonian operator from below. It will become explicit below that it is precisely the modes with imaginary λ that make the Hamiltonian unbounded, as otherwise was to be expected. Hence not all the states to be constructed in this section are within reach for an observer but only those with positive eigenenergies.

It is convenient to split H as the sum

$$H = H_{\text{flat}} + H_{pp}$$

of a contribution

$$H_{\text{flat}} = \frac{1}{4\pi\alpha'}\int_0^\pi d\sigma\,\big[\big(\partial_\tau X^a\big)^2 + \big(\partial_\sigma X^a\big)^2\big]$$

from the flat a-directions and a contribution

$$H_{pp} = \frac{1}{4\pi\alpha'}\int_0^\pi d\sigma\,\big\{\big(\partial_\tau X^i\big)^2 + \big(\partial_\sigma X^i\big)^2 - m^2\kappa^2\big[\big(X^1\big)^2 - \big(X^2\big)^2\big]\big\}$$

from the 1, 2-directions. The eigenstates of H are then of the form $|\psi\rangle = |\psi_{\text{flat}}\rangle \otimes |\psi_{pp}\rangle$ and the eigenenergies read $E = E_{\text{flat}} + E_{pp}$, with $\{|\psi_{\text{flat}}\rangle, E_{\text{flat}}\}$ and $\{|\psi_{pp}\rangle, E_{pp}\}$ the solutions to the eigenvalue problems

$$H_{\text{flat}}|\psi_{\text{flat}}\rangle = E_{\text{flat}}|\psi_{\text{flat}}\rangle,$$
$$H_{pp}|\psi_{pp}\rangle = E_{pp}|\psi_{pp}\rangle.$$

5.1. Eigenvalue problem for H_{flat}

Apart from the number of dimensions, it is the same problem as for the open string in Minkowski space–time. Using the mode expansions for X^a, one obtains for H_{flat} a sum

$$H_{\text{flat}} = \frac{1}{2\alpha'}\sum_{n=1}^{\infty} {:}c_n^{a\dagger} c_n^a{:} + \alpha' p_a^2 + \frac{D-4}{24}$$

of harmonic oscillator Hamiltonians, one for every frequency $n > 0$ in every direction a. As usual, $:AB:$ denotes normal ordering of AB and the sum $\sum_{n>0} n$ entering the normal ordering constant has been regulated using ζ-regularization, so that it takes the value $\zeta(-1) = -1/12$. The solution for H_{flat} is well known. The Fock space is formed by states

$$|\psi_{\text{flat}}\rangle = |\psi_{\{k_n^a\}}\rangle = \bigotimes_{a=3}^{D-2} \big|\big\{k_n^a\big\}_{n=1}^{\infty}, p_a\big\rangle, \quad k_n^a = 0, 1, 2, \ldots, \tag{5.1}$$

with k_n^a the occupancy number of the harmonic oscillator of frequency n in the a-direction. The energies of these states are

$$E_{\text{flat}} = E_{\{k_n^a\}} = \sum_{a=3}^{D-2} \sum_{n=1}^{\infty} n k_n^a + \alpha' p_a^2 + \frac{D-4}{24}.$$

We note that the sum over n is actually finite, since for every eigenstate there is a finite number of non-zero occupancy numbers k_n^a. The action of $c_n^{b\dagger}$ and c_n^b on $|\{k_r^a\}, p_a\rangle$ is $\sqrt{2\alpha' n}$ times the usual one of creation and annihilation harmonic oscillator operators.

5.2. *Eigenvalue problem for H_{pp}*

Employing the mode expansions for X^i, we obtain after some work that

$$H_{pp} = H_{\Lambda_\pm} + \delta_{m^2\kappa^2,\text{even}} H_{\{(k,l)\}} + \delta_{m\kappa,\text{even}} H_{\text{e}} + \delta_{m\kappa,\text{odd}} H_{\text{o}}, \tag{5.2}$$

where $H_{\Lambda_\pm}$ is given by

$$H_{\Lambda_\pm} = \frac{1}{2\pi\alpha'} \sum_{\lambda\in\Lambda_\pm} \lambda f(\lambda) c_\lambda c_{-\lambda} \tag{5.3}$$

and $H_{\{(k,l)\}} H_{\text{e}}$ and H_{o} take the form

$$H_{\{(k,l)\}} = \frac{1}{4\alpha'} \sum_{\lambda(k,l)} \lambda\big[f_a(\lambda) a_\lambda a_{-\lambda} + f_b(\lambda) b_\lambda b_{-\lambda}\big], \tag{5.4}$$

$$H_{\text{e,o}} = \frac{1}{4\pi\alpha'} \left[\frac{\pi}{4}\cosh(m\kappa\pi) + \frac{B^2}{m\kappa}\sinh(m\kappa\pi) - 1\right] b_{\text{e,o}}^2. \tag{5.5}$$

We first study the problem for $H_{\Lambda_\pm}$ and postpone the solution for the pathological modes $m^2\kappa^2 =$ even, $m\kappa =$ even, $m\kappa =$ odd.

We recall that an infinite number of the modes $\lambda \in \Lambda_\pm$ have real λ and that a finite number of them have imaginary λ. We separate their contributions H_{R} and H_{I} to $H_{\Lambda_\pm}$ and write

$$H_{\Lambda_\pm} = H_{\text{R}} + H_{\text{I}}.$$

The eigenstates and eigenvalues of $H_{\Lambda_\pm}$ are $|\psi_{\Lambda_\pm}\rangle = |\psi_{\text{R}}\rangle \otimes |\psi_{\text{I}}\rangle$ and $E_{\Lambda_\pm} = E_{\text{R}} + E_{\text{I}}$, with $\{|\psi_{\text{R}}\rangle, E_{\text{R}}\}$ and $\{|\psi_{\text{I}}\rangle, E_{\text{I}}\}$ solutions to the problems

$$H_{\text{R}}|\psi_{\text{R}}\rangle = E_{\text{R}}|\psi_{\text{R}}\rangle,$$
$$H_{\text{I}}|\psi_{\text{I}}\rangle = E_{\text{I}}|\psi_{\text{I}}\rangle.$$

5.2.1. *Solution for H_{R}*

The commutation relations (3.15) for the operators c_λ, yield for H_{R}

$$H_{\text{R}} = \frac{1}{\pi\alpha'} \sum_{\substack{\lambda\in\Lambda_\pm \\ \operatorname{Re}\lambda>0}} \lambda f(\lambda){:}c_\lambda^\dagger c_\lambda{:} + K_{\text{R}},$$

where K_{R} is the normal ordering constant

$$K_{\mathrm{R}} = -\frac{1}{2} \sum_{\substack{\lambda\in\Lambda_\pm \\ \operatorname{Re}\lambda>0}} \lambda. \tag{5.6}$$

The Hamiltonian H_{R} is a sum of harmonic oscillators, one for every real $\lambda > 0$. The eigenstates of H_{R} are then harmonic oscillator states

$$|\psi_{\mathrm{R}}\rangle = \big|\{k_\lambda\}_{\operatorname{Re}\lambda>0}\big\rangle, \quad k_\lambda = 0, 1, 2, \ldots, \tag{5.7}$$

with k_λ the occupancy number for the harmonic oscillator of frequency λ, while the eigenenergies read

$$E_{\mathrm{R}} = E_{\{k_\lambda, \operatorname{Re}\lambda>0\}} = \sum_{\operatorname{Re}\lambda>0} \lambda k_\lambda + K_{\mathrm{R}}.$$

The action of $c_\lambda^\dagger$ and c_λ on the states $|\{k_{\lambda'}\}\rangle$ is $\sqrt{\pi\alpha'/f(\lambda)}$ times the usual action of annihilation and creation harmonic oscillator operators.

Since there are infinitely many positive real λ with no accumulation point, the normal ordering constant K_{R} needs regularization. For every $m\kappa$ we can always take a sufficiently large integer N such that $m^2\kappa^2 \ll N^2$ and split the sum for K_{R} into two sums: one over $0 < \lambda < N$ and one over $N < \lambda$. Since $(m\kappa/N)^2 \ll 1$, the λ's in the second sum are given by Eq. (2.42), so that K_{R} can be written as

$$K_{\mathrm{R}} = -\frac{1}{2} \sum_{\operatorname{Re}\lambda<N} \lambda + \frac{1}{2}\sum_{n=1}^{N} (\lambda_n + \tilde{\lambda}_n) - \frac{1}{2}\sum_{n=1}^{\infty} (\lambda_n + \tilde{\lambda}_n).$$

The first two terms in this equation are finite, while accordingly to (2.42) the third one contains the divergent sum $\sum_{n>0} n$. Regularizing this in the same way as for the flat a-directions we arrive at

$$K_{\mathrm{R}} = \frac{1}{12} + \Delta K(m),$$

where $\Delta K(m)$ collects all m-dependent contributions to K_{R}. For example, for $m^2\kappa^2 \ll 1$ the integer N can be taken equal to 1 and from Section 2 it is straightforward to see that

$$\Delta K\big(m^2\kappa^2 \ll 1\big) = -\frac{m\kappa}{2\sqrt{1+B^2}} \left[1 - \frac{(m\kappa)^2}{12} \frac{\pi^2 B^2}{1+B^2} + \mathcal{O}\big(m^3\kappa^3\big)\right].$$

5.2.2. Solution for H_{I}

It is convenient to introduce for every imaginary λ operators $\hat{q}_\lambda$ and $\hat{p}_\lambda$ defined by

$$c_{\pm\lambda} = \sqrt{\frac{\pi\alpha'}{2|\lambda f(\lambda)|}} (\hat{q}_\lambda \pm \hat{p}_\lambda), \quad \operatorname{Im}\lambda > 0. \tag{5.8}$$

They are Hermitean and satisfy commutation relations $[\hat{q}_\lambda, \hat{p}_\lambda] = i\,\mathrm{sign}[\lambda f(\lambda)]$. In terms of them, H_{I} takes the form

$$H_{\mathrm{I}} = \sum_{\substack{\lambda\in\Lambda_\pm \\ \operatorname{Im}\lambda>0}} \mathrm{sign}\big[\lambda f(\lambda)\big]\big(\hat{p}_\lambda^2 - \hat{q}_\lambda^2\big).$$

It is clear that the $H_{\rm I}$ is not bounded from below. Let us forget for a moment about this and formally solve the eigenvalue problem for $H_{\rm I}$. The solution is given by $|\psi_{\rm I}\rangle = \prod |\varphi_\lambda\rangle$ and $E_{\rm I} = \sum E_\lambda$, with the product and the sum extended over all imaginary λ with $\operatorname{Im}\lambda > 0$, and $\{|\psi_\lambda\rangle, E_\lambda\}$ being solutions of

$$\big(\hat{p}_\lambda^2 - \hat{q}_\lambda^2\big)|\psi_\lambda\rangle = E_\lambda|\psi_\lambda\rangle, \quad \operatorname{Im}\lambda > 0. \tag{5.9}$$

To solve (5.9) we work in a position representation, in which the wave function for $|\psi_\lambda\rangle$ is $\psi_\lambda(q_\lambda)$ and the operators $\hat{q}_\lambda$ and $\hat{p}_\lambda$ act on it through multiplication and derivation, i.e., $\hat{q}_\lambda \to q_\lambda$ and $\hat{p}_\lambda \to i\frac{d}{dq_\lambda}$. Eq. (5.9) then becomes

$$\left(\frac{d^2}{dq_\lambda^2} + q_\lambda^2 + E_\lambda\right)\psi_\lambda(q_\lambda) = 0, \quad \operatorname{Im}\lambda > 0.$$

This is the time-independent Schrödinger equation for a particle in an inverted harmonic potential. Such equation does not have bound states and for every real E_λ admits

$$\psi_{\lambda,1}(q_\lambda) = e^{-iq_\lambda^2/2} q_\lambda \Phi\left(\frac{3}{4} + \frac{iE}{4}, \frac{3}{2}; iq_\lambda^2\right)$$

and

$$\psi_{\lambda,2}(q_\lambda) = e^{-iq_\lambda^2/2} \Phi\left(\frac{1}{4} + \frac{iE}{4}, \frac{1}{2}; iq_\lambda^2\right)$$

as two linearly independent solutions, $\Phi(\mu,\nu;z)$ being the degenerate hypergeometric function. Both $\psi_{\lambda,1}(q_\lambda)$ and $\psi_{\lambda,2}(q_\lambda)$ are regular at $q_\lambda = 0$, while at $|q_\lambda| \to \infty$ are superpositions of the oscillating exponentials

$$\frac{1}{\sqrt{|q_\lambda|}} \exp\left[\pm\frac{i}{4}\big(E_\lambda \ln q_\lambda^2 + 2q_\lambda^2\big)\right].$$

The most general solution for $\psi_\lambda(q_\lambda)$ is then an arbitrary linear combination

$$\psi_\lambda(q_\lambda) = C_1\psi_{\lambda,1}(q_\lambda) + C_2\psi_{\lambda,2}(q_\lambda).$$

The state $\psi_\lambda(q_\lambda)e^{iE_\lambda\tau}$ is a scattering state which in this position representation is asymptotically formed by one incoming and one outgoing traveling wave. It is worth noting that these waves are not plane and that the effect of the inverted harmonic potential is felt at $|q_\lambda| \to \infty$. The eigenstates of $H_{\rm I}$ are then

$$|\psi_{\rm I}\rangle = \big|\{E_\lambda\}_{\operatorname{Im}\lambda>0}\big\rangle \to \prod_{\operatorname{Im}\lambda>0} \psi_\lambda(q_\lambda), \quad E_\lambda \text{ real and arbitrary}, \tag{5.10}$$

and the energies read

$$E_{\rm I} = \sum_{\operatorname{Im}\lambda>0} \operatorname{sign}\big[\lambda f(\lambda)\big] E_\lambda.$$

The action of $c_{\pm\lambda}$ on $\psi_\lambda(q_\lambda)$ is through (5.8) and multiplication and derivation. The states $|\psi_{\rm I}\rangle$ play in the 1 and 2-directions the equivalent rôle to that of the plane wave states $|p_a\rangle$ in the flat a-directions. One way to ensure that the eigenenergies are non-negative is to restrict to scattering states with $E_\lambda = \operatorname{sign}[\lambda f(\lambda)]|E_\lambda|$ for every imaginary λ.

Putting everything together, the eigenstates and eigenvalues of $H_{\Lambda_\pm}$ are

$$|\psi_{\Lambda_\pm}\rangle = |\psi_{\{k_n^a\}}\rangle \otimes \big|\{k_\lambda\}_{\mathrm{Re}\,\lambda>0}\big\rangle \otimes \big|\{E_\lambda\}_{\mathrm{Im}\,\lambda>0}\big\rangle,$$
$$E_{\Lambda_\pm} = \sum_{a=3}^{D-2}\sum_{n=1}^{\infty} n k_n^a + \sum_{\mathrm{Re}\,\lambda>0} \lambda k_\lambda + \alpha' p_a^2 + \sum_{\mathrm{Im}\,\lambda>0} \mathrm{sign}\big[\lambda f(\lambda)\big] E_\lambda + \frac{D-2}{24} + \Delta K(m).$$

5.2.3. Contributions from $H_{\{(k,l)\}}$, H_{e}, H_{o}

If $m\kappa$ is such that it squares to an even integer, or is itself an even or odd integer, the Hamiltonian also receives the contributions $H_{\{(k,l)\}}$, H_{e} and H_{o} in (5.4)–(5.5). In case $m^2\kappa^2 = \text{even}$, it is trivial to see that

$$H_{\{(k,l)\}} = \frac{1}{2\alpha'} \sum_{\lambda(k,l)>0} \lambda\big[f_a(\lambda) a_\lambda^\dagger a_\lambda + f_a(\lambda) b_\lambda^\dagger b_\lambda\big] + \sum_{\lambda(k,l)>0} \lambda.$$

This only adds to the total Hamiltonian two harmonic oscillators for every $\lambda(k,l)$, one for the a_λ-mode and one for the b_λ-mode, and contributes to the normal ordering constant with a finite quantity. The eigenstates and eigenenergies are trivial to write. Assume for example $m^2\kappa^2 = 6$. There is then only one solution for (k,l), namely $k = 2, l = 1$ and $\lambda = \sqrt{10}$. This adds two oscillators to the total Hamiltonian and $\sqrt{10}$ to the normal ordering constant.

For $m\kappa = \text{even}$ and $m\kappa = \text{odd}$, the Hamiltonian adds an m-dependent momentum-like contribution to the energy.

6. Conclusion and outlook

In this paper we have canonically quantized the open string on the Penrose limit of $dS_n \times S^n$ supported by constant antisymmetric B_2 and a constant dilaton. Canonical quantization has proved perfectly suited for the task, thus making unnecessary to resort to Dirac quantization and avoiding the problem of whether the boundary conditions for the string endpoints should be regarded as first or second class constraints. The position operators for the quantized string define non-commutative spaces, the wave fronts, for all values of the string parameter σ. Noticeably non-commutativity is not restricted to the string endpoints but extends outside the brane on which the endpoints may be assumed to move. The Minkowski limit is smooth and reproduces the results in the literature [19].

We think that further investigation of strings on plane-wave backgrounds is worth to understand non-commutativity in relation with gravity. The low-energy field-theory limit looks particularly interesting since it may shed light on an effective theory for non-commutative gravity. It must be mentioned in this regard that there is a vast literature [30] on the formulation of Seiberg–Witten maps for gravity and effective non-commutative corrections to general relativity solutions, plane waves among them [31].

From a purely string theory point of view, the strings considered here may be thought of as "in" or "out" states to study string scattering on more complicated spaces, which in turn will have a Penrose limit, and strings near space–time singularities [32].

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Appendix A. Explicit expression for the string momentum

We collect here the contributions to the string momentum components $P_i(\tau,\sigma)$ in Eq. (2.37) of the various existing modes. They are obtained by using (2.11) for X^i_{odd}, X^i_{even}, $X^i_{(k,l)}$ and X^i_λ. For the modes X^i_{o} and X^i_{e}, in (2.16)–(2.17) and (2.18)–(2.19), we have

$$2\pi\alpha' P_{1,\mathrm{o}} = -\frac{m\kappa}{B} b_{\mathrm{o}} \left\{ \sinh\left(\frac{m\kappa\pi}{2}\right)\cos(m\kappa\sigma) - B^2 \sinh\left[m\kappa\left(\frac{\pi}{2}-\sigma\right)\right]\right\},$$

$$2\pi\alpha' P_{2,\mathrm{o}} = -m\kappa B\left[a_{\mathrm{o}} - \frac{m\kappa}{B} b_{\mathrm{o}}\tau \sinh\left(\frac{m\kappa\pi}{2}\right)\right]\sin(m\kappa\sigma)$$

and

$$2\pi\alpha' P_{1,\mathrm{e}} = \frac{m\kappa}{B} b_{\mathrm{e}} \left\{ \cosh\left(\frac{m\kappa\pi}{2}\right)\cos(m\kappa\sigma) + B^2 \cosh\left[m\kappa\left(\frac{\pi}{2}-\sigma\right)\right]\right\},$$

$$2\pi\alpha' P_{2,\mathrm{e}} = -m\kappa B\left[a_{\mathrm{e}} + \frac{m\kappa}{B} b_{\mathrm{e}}\tau \cosh\left(\frac{m\kappa\pi}{2}\right)\right]\sin(m\kappa\sigma).$$

The contribution of the modes $X^i_{(k,l)}$ in (2.22)–(2.23) in turn reads

$$2\pi\alpha' P_{1,(k,l)} = \left[\frac{\alpha}{B} a_{\lambda(k,l)}\big(\cos\beta\sigma + B^2\cos\alpha\sigma\big) + b_{\lambda(k,l)}\left(\sin\beta\sigma - \frac{\alpha\beta}{\lambda^2}\sin\alpha\sigma\right)\right] e^{-i\lambda\tau},$$

$$\begin{aligned} 2\pi\alpha' P_{2,(k,l)} = i\lambda\bigg[& a_{\lambda(k,l)}\left(\sin\alpha\sigma - \frac{\alpha\beta}{\lambda^2}\sin\beta\sigma\right) \\ & + \frac{\beta}{B\lambda^2} b_{\lambda(k,l)}\big(\cos\alpha\sigma + B^2\cos\beta\sigma\big)\bigg] e^{-i\lambda\tau}. \end{aligned}$$

Finally, the modes X^i_λ in (2.33)–(2.35) yield the contributions

$$\begin{aligned} 2\pi\alpha' P_{1,\pm}(\tau,\sigma) = c_\lambda \frac{\alpha}{B}\bigg[& \cos\beta\sigma + \frac{\sin\beta\pi}{\cos\beta\pi \mp 1}\sin\beta\sigma \\ & - B^2\left(\frac{\cos\alpha\pi \pm 1}{\sin\alpha\pi}\sin\alpha\sigma - \cos\alpha\sigma\right)\bigg] e^{-i\lambda\tau} \end{aligned}$$

and

$$\begin{aligned} 2\pi\alpha' P_{2,\pm}(\tau,\sigma) = i c_\lambda \lambda\bigg[& \frac{\cos\alpha\pi \pm 1}{\sin\alpha\pi}\cos\alpha\sigma + \sin\alpha\sigma \\ & - \frac{\alpha\beta}{\lambda^2}\left(\sin\beta\sigma - \frac{\sin\beta\pi}{\cos\beta\pi \mp 1}\cos\beta\sigma\right)\bigg] e^{-i\lambda\tau}. \end{aligned} \tag{A.1}$$

Appendix B. Derivation of Eqs. (4.6)–(4.7)

Organizing the modes in the four sets Λ^{I}, Λ^{R}, Λ, $\tilde{\Lambda}$ introduced in Section 3 and expanding (4.2) in powers of $m^2\kappa^2 \ll 1$, the function $\Theta(\sigma,\sigma')$ becomes a sum

$$i\Theta(\sigma,\sigma') = \sum_{k=I,R,\Lambda,\tilde{\Lambda}} i\Theta(k;\sigma,\sigma')$$

of four contributions $\Theta(k;\sigma,\sigma')$, each one of which is a power series in $m\kappa$. Up to order four in $m\kappa$, these contributions read

$$i\Theta(\mathrm{I};\sigma,\sigma') = \frac{i\alpha' B}{2(1+B^2)}(\pi-2\sigma) - \frac{i\alpha' B(m\kappa)^2}{12(1+B^2)^2}\big[\sigma(\sigma-\pi)\big(2+B^2\big) - 3B^2\sigma'(\sigma'-\pi)-\pi^2\big] + \mathcal{O}\big(m^4\kappa^4\big), \tag{B.1}$$

$$i\Theta(\mathrm{R};\sigma,\sigma') = \frac{i\alpha' B}{2(1+B^2)}(\pi-2\sigma') + \frac{i\alpha' B(m\kappa)^2}{12(1+B^2)^2}\big[\sigma'(\sigma'-\pi)\big(2+B^2\big) - 3B^2\sigma(\sigma-\pi)-\pi^2\big] + \mathcal{O}\big(m^4\kappa^4\big), \tag{B.2}$$

$$i\Theta(\Lambda;\sigma,\sigma') = -\frac{2i\alpha' B}{1+B^2}\sum_{n=1}^{\infty}\frac{\cos n\sigma\,\sin n\sigma'}{n} + \frac{i\alpha' B(m\kappa)^2}{(1+B^2)^2} \times\sum_{n=1}^{\infty}\bigg[B^2(2\sigma-\pi)\frac{\sin n\sigma\,\sin n\sigma'}{n^2} + (2\sigma'-\pi)\frac{\cos n\sigma\,\cos n\sigma'}{n^2} - 2\big(1-B^2\big)\frac{\cos n\sigma\,\sin n\sigma'}{n^3}\bigg] + \mathcal{O}\big(m^4\kappa^4\big) \tag{B.3}$$

and

$$i\Theta(\tilde{\Lambda};\sigma,\sigma') = -\frac{2i\alpha' B}{1+B^2}\sum_{n=1}^{\infty}\frac{\sin n\sigma\,\cos n\sigma'}{n} - \frac{3i\alpha' B(m\kappa)^2}{(1+B^2)^2}\sum_{n=1}^{\infty}\frac{\sin n\sigma\,\cos n\sigma'}{n^3} + \mathcal{O}\big(m^4\kappa^4\big). \tag{B.4}$$

Summing all the contributions of order zero in $m\kappa$ in these equations, we have

$$i\Theta_0(\sigma,\sigma') = \frac{i\alpha' B}{1+B^2}\big[\pi-\sigma_+ - F_1(\sigma_+)\big],$$

where $F_1(\sigma_+)$ is the Fourier series (3.28). This trivially leads to the order zero contributions in Eqs. (4.6) and (4.7). To sum the order two contributions, we first note that

$$\big[i\Theta(\Lambda)+i\Theta(\tilde{\Lambda})\big]_2 = -\frac{i\alpha' B}{4(1+B^2)^2}\big\{\big[B^2(\sigma_+ +\sigma_- -\pi) + (\sigma_+ -\sigma_- -\pi)\big]F_2(\sigma_-) - \big[B^2(\sigma_+ +\sigma_- -\pi) - (\sigma_+ -\sigma_- -\pi)\big]F_2(\sigma_+) + \big(2B^2+1\big)F_3(\sigma_+) + \big(5-2B^2\big)F_3(\sigma_-)\big\}, \tag{B.5}$$

where the Fourier series $F_2(\sigma_\pm)$ are as in (3.29)–(3.30) and $F_3(\sigma_\pm)$ read

$$F_3(\sigma_-) := 2\sum_{n=1}^{\infty}\frac{\sin n\sigma_-}{n^3} = \frac{\sigma_-^3}{6} - \frac{\pi}{2}\sigma_-|\sigma_-| + \frac{\pi^2}{3}\sigma_-,$$

$$F_3(\sigma_+) := 2\sum_{n=1}^{\infty}\frac{\sin n\sigma_+}{n^3} = \frac{\sigma_+^3}{6} - \frac{\pi}{2}\sigma_+^2 + \frac{\pi^2}{3}\sigma_+.$$

Eqs. (B.1), (B.2) and (B.5) then lead to the second order contributions in Section 5.

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